NEARLY NORMAL EXPANSION FOR
FORCED AND INHOMOGENEOUS TURBULENCE

W. C. Meecham and D. -t. Jeng

31

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PREFACE

Turbulent and fluid-flow phenomena have an important influence on atmospheric and oceanic circulation, and are hence expected to be of considerable concern in modeling weather situations, and estimating the consequence of weather-modification attempts. For reasons of simplifying an already very complicated problem, past theoretical turbulence work has been largely restricted to an idealization called homogeneous turbulence. In this report an effort is made to extend one kind of turbulence theory, a theory for "nearly Gaussian" processes, to the more general and more practical class of problems involving nonhomogeneous flows.

An equilibrium energy spectrum for the turbulence in such problems is found. It is one which approximates many experiments reasonably well. A hypothesis for one term in the theory is proposed, and the present theory is used to correct the empirical "mixing length" theory of turbulence.

Dr. Meecham performed this study as a consultant to The RAND Corporation. He and Dr. D. -t. Jeng are associated with The University of California, Los Angeles.

v

ABSTRACT

Earlier work with the nearly-normal expansion is extended to include forced, statistically stationary turbulence. The nature of the time-transformation of the Cameron-Martin-Wiener representation is discussed. For a stationary process we obtain much information using the representation at one time. For real turbulence we find $E(k) \sim k^{-2}$; this spectrum is discussed and shown to be related to turbulence intermittency. Using a hypothesis for the (small) non-Gaussian part of the velocity -- a hypothesis suggested by earlier work -- a corrected mixing-length theory is found with an eddy viscosity: $\nu_D = u'l(a_1 - a_2 k_0^{-2} \nabla^2)$ with a_1, a_2 dimensionless, positive and of the same order; k_0 the wave number of the spectrum maximum, and u' and l the rms velocity fluctuation and the turbulence correlation length respectively.

ACKNOWLEDGMENTS

CONTENTS

I. INTRODUCTION

In recent work we have described the use of nearly-normal expansions (based on the work of Cameron and Martin, and that of Wiener)[1,2] for the representation of turbulence. The expansions were used to deal with problems of decaying turbulence, modeled and real.[3-5] In Ref. 3 the correct equilibrium energy spectrum $[E(k) \sim k^{-2}]$ was obtained for Burgers' model turbulence. To do this, a hypothesis was used that concerns the nature of the non-Gaussian term in the velocity field. The same result can be obtained, without hypothesis, for a stationary Burgers' process.[6] It is known that the Cameron-Martin-Wiener expansion is not uniform in time for nonforced processes.[6-8] Thus, although the expansion may converge rapidly initially, at later times the convergence becomes less rapid even for nearly normal processes. Nevertheless, it has been found that the convergence is adequate for the treatment of real wind-tunnel turbulence, for up to six correlation times (or 120 mesh lengths downstream from the wind-tunnel grid).[5] The expansion has certain virtues not possessed by some other representations for turbulence. These advantages have been emphasized in earlier work and will not be recalled here.

In most practical problems involving turbulence, one is faced with statistically inhomogeneous turbulence, which is frequently driven and approximately in statistical equilibrium. The driving source is very often found in large-scale effects deriving their energy from the mean flow. For example, we point to the existence of turbulence in the lower atmosphere which is driven by wind shear in that region. Another important application is in turbulent wakes, for instance of re-entering vehicles. In such applications we find the turbulence driven by large eddies which couple into the mean shear flow and derive their energy from that flow.

We shall here replace the large-scale energy sources by an equivalent forcing term. We suppose that the process has existed long enough that it is statistically stationary (in a statistically inhomogeneous problem the forcing term may vary with position). It is easy to generalize the random process expansion for such problems (see Ref. 5, Section

II). Recently, Saffman[11] has used this kind of stationary expansion to deal with turbulent diffusion, and has obtained some very promising results in this way. We formulate the problem for stationary, incompressible real turbulence and obtain the equilibrium spectrum in that case. The result is k^{-2} in the inertial range. Dryden[13] has observed approximately such spectral behavior for turbulence at moderate Reynolds' number. This spectrum has been suggested by Townsend, using a kinematical argument (the discussion here gives the results using the dynamics).

The flow realizations leading to various spectral laws will be discussed. In particular, as is known, k^{-2} is characteristic of flows with near discontinuities, or intermittences.

A hypothesis is proposed concerning the form of the non-Gaussian term in the velocity-field representation. From this hypothesis the inertial transfer is obtained and a correction to mixing-length theory is proposed.

II. DRIVEN TURBULENCE

We develop the equations for stationary turbulence in an incompressible fluid, relying on previous work for some of the details.[3-5] Define the Fourier transform of the velocity field

$$u_i(\underline{k},t) = \int e^{i\underline{k}\cdot\underline{r}} u_i(\underline{r},t)d\underline{r}$$

(and similarly for f). The Fourier transform of the incompressible Navier-Stokes equation is

$$\left(\frac{\partial}{\partial t} + \nu k^2\right)u_i(\underline{k}) = \frac{1}{2}(2\pi)^{-3}P_{ij\ell}(\underline{k})\int u_j(\underline{k} - \underline{k}')u_\ell(\underline{k}')d\underline{k}' + f_i(\underline{k},t)$$

$$(2.1)$$

with

$$k_i u_i = 0$$

and

$$P_{ij\ell}(\underline{k}) = k_\ell P_{ij}(\underline{k}) + k_j P_{i\ell}(\underline{k})$$

and

$$P_{ij}(\underline{k}) = \delta_{ij} - k^{-2}k_i k_j .$$

The given forcing term $\underline{f}$ will be assumed to be statistically homogeneous and Gaussian and confined to large-scale effects. That is, it is supposed that $\underline{f}$ vanishes for k larger than an energy-containing wave number k_0. The first two random processes in the Cameron-Martin-Wiener representation are

$$_t H_i^{(1)}(\underline{r}, t_1)$$

$$_t H_{ij}^{(2)}(\underline{r}_1, \underline{r}_2; t_1, t_2) \equiv {}_t H_i^{(1)}(\underline{r}_1, t_1) \; {}_t H_j^{(1)}(\underline{r}_2, t_2) - \delta_{ij} \, \delta(\underline{r}_1 - \underline{r}_2) \delta(t_1 - t_2)$$

$$(2.2)$$

with the covariance property

$$\langle \; {}_t H_i^{(1)}(\underline{r}_1, t_1) \; {}_t H_j^{(1)}(\underline{r}_2, t_2) \rangle = \delta_{ij} \, \delta(\underline{r}_1 - \underline{r}_2) \delta(t_1 - t_2) \; .$$

The presubscript t indicates a transformation (as yet unspecified) in the basic white-noise process. We discuss these transformations below.

We shall suppose later that the turbulence is statistically stationary (it is known that certain necessary conditions must be imposed for stationarity);[12] and that it is (t least locally) statistically homogeneous and isotropic. We suppose con·ergence sufficiently rapid that the random field can be adequately represented by just the first two terms in the nearly normal expansion at a given instant. Arguments have been presented for the near Gaussianity of energy-range turbulence.[5] We suppose the process to be nearly normal in the inertial range as well. We have

$$u_i(\underline{r}, t) \equiv u_i^{(1)} + u_i^{(2)}$$

$$= \iint K_{i\alpha}^{(1)}(\underline{r} - \underline{r}'; t, t') \; {}_t H_\alpha^{(1)}(\underline{r}', t') d\underline{r}' dt'$$

$$+ \iiiint K_{i\alpha\beta}^{(2)}(\underline{r} - \underline{r}_1, \; \underline{r} - \underline{r}_2; t, t_1, t_2)$$

$$\times {}_t H_{\alpha\beta}^{(2)}(\underline{r}_1, \underline{r}_2; t_1, t_2) d\underline{r}_1 d\underline{r}_2 dt_1 dt_2 \; . \qquad (2.3)$$

$$f_i(\underline{r},t) = \iint F_{i\alpha}^{(1)}(\underline{r} - \underline{r}'; t,t') \, _tH_\alpha^{(1)}(r',t')d\underline{r}' \, dt'$$

$$+ \iiiint F_{i\alpha\beta}^{(2)} (\underline{r} - \underline{r}_1, \underline{r} - \underline{r}_2; t,t_1,t_2)$$

$$\times \, _tH_{\alpha\beta}^{(2)}(\underline{r}_1,\underline{r}_2; t_1,t_2)d\underline{r}_1 d\underline{r}_2 dt_1 dt_2 \qquad (2.4)$$

We suppose that our representation is such that we can use the manifestly statistically homogeneous form involving kernels that are spatial functions of the difference variable. The transformation of the basis in time represented by $_tH^{(1)}$ is assumed to have this characteristic.

Consider the behavior of the Cameron-Martin-Wiener (C-M-W) representation in time. It is known[6-8] that the relation of the random process to a given representation changes in time, in many nonlinear dynamical problems, in a way analogous to a relative rotation of a vector with respect to its coordinate system. Thus if we consider a nearly normal process, u_i, at some instant of time, we may think of the first term in the representation [i.e., the first term of (2.3)] as the projection of the process on the "Gaussian" axis, and similarly for the second term on the lowest-order "non-Gaussian" axis. Then as time goes on, although the process may remain nearly normal, there is a rotation of the process relative to the axes, giving different components at later times on the various axes. In this language the averages (the moments) correspond to vector-dot products of the various quantities, and these products are independent of the coordinate system used in their calculation. The moments are correctly given by any representation that is complete, and the one used here is complete, so we can always expect correct results, even at later times when the process has transformed. But of course it may be necessary to use more terms in the representation as time goes on. There is a "best" representation for a nearly normal process at any time, which yields a maximum value for the projection on the "Gaussian" axis, but there are special difficulties for an effort directed toward finding this "best" representation.[9-10] This

kind of transformation of the representation can occur even for stationary processes. Thus, although at a given time the stationary process may be representable by the first term in a nearly normal expansion, in general at a later time higher-order terms will be needed. Furthermore for the specified stationary Gaussian force, $\underline{f}$, we could at any instant use one term; but the "best" velocity-field representation is transforming, and since we must use the same representation for the two, we must allow the possibility of transformation even for the force. This reflects in the requirement that the kernel for the force in (2.4) must be a separate function of the times t and t', and not a function of just the difference. At a given time, say t = 0, we can for a Gaussian force and a homogeneous and stationary process, choose a "best" representation. The force then has only the first term of (2.4) and all kernels are functions of the space difference variables. Our work here will be mainly concerned with the use of such a fixed representation.

Returning to (2.1) we construct the usual energy equation. Multiply by the velocity at a second wavenumber (and the same time), symmetrize and average to find,

$$\left[\frac{\partial}{\partial t} + \nu(k_1^2 + k_2^2)\right] \langle u_i(\underline{k}_1)u_j(\underline{k}_2)\rangle = \frac{i}{2}(2\pi)^{-3} P_{i\alpha\beta}(\underline{k}_1)$$

$$\times \int \langle u_\alpha(\underline{k}_1 - \underline{k}')u_\beta(\underline{k}')u_j(\underline{k}_2)\rangle d\underline{k}'$$

$$+ \langle f_i(\underline{k}_1)u_j(\underline{k}_2)\rangle$$

$$+ \text{ same, interchange subscripts i,j}$$
$$\text{and variables } \underline{k}_1, \underline{k}_2$$

$$(2.5)$$

For homogeneous processes we know that

$$\langle u_i(\underline{k}_1)u_j(\underline{k}_2)\rangle \sim V, \quad \underline{k}_1 + \underline{k}_2 = 0$$

$$\sim 0, \text{ otherwise}$$

$$(2.6)$$

where V is the volume of the turbulence. There are similar relations for other moments.

Using the usual definitions[14] we have

$$E(k) = \left[(2\pi)^2 V\right]^{-1} k^2 \langle u_i(\underline{k})u_i^*(\underline{k})\rangle \tag{2.7}$$

and for isotropic processes (2.6) becomes

$$\left(\frac{\partial}{\partial t} + 2\nu k^2\right) E(k) = T(k) + S(k) \tag{2.8}$$

where

$$T(k) = \frac{i}{2V}(2\pi)^{-5} k^2 P_{i\alpha\beta}(\underline{k}) \int \langle u_\alpha(\underline{k} - \underline{k}')u_\beta(k')u_i(-\underline{k})\rangle d\underline{k}' + c.c. \tag{2.9}$$

$$S(k) = \langle f_i(\underline{k})u_i^*(\underline{k})\rangle + c.c. \tag{2.10}$$

(Actually $\dot{E}$ vanishes for stationary processes.) Here c.c. stands for the complex conjugate of a preceeding term.

From the properties of the representation (2.3) we know that to lowest order in $\underline{u}^{(2)}$, the transfer can be written

$$T(k) = \frac{i}{2V}(2\pi)^{-5} k^2 P_{i\alpha\beta}(\underline{k}) \int \langle 2u_\alpha^{(1)}(\underline{k} - \underline{k}')u_\beta^{(2)}(\underline{k}')u_i^{(1)}(-\underline{k})$$

$$\tag{2.11}$$

$$+ u_\alpha^{(1)}(\underline{k} - \underline{k}')u_\beta^{(1)}(\underline{k}')u_i^{(2)}(-\underline{k})\rangle d\underline{k}' + c.c.$$

where $\underline{u}^{(j)}(\underline{k})$ are the Fourier transforms of $\underline{u}^{(j)}(\underline{r})$, time dependence implicit. To write the transfer, we need information about the non-Gaussian term. A hypothesis for $\underline{u}^{(2)}$ is discussed below. $T(k) = 0$ in the inertial range, i.e., for $k_0 < k <$ viscous wave number, when the process is in equilibrium.

For a stationary process $\dot{T}(k) = 0$ for all k. This gives us a condition on the (dominant) Gaussian term alone. If we take the derivation

of (2.9) and use (2.1) to replace $\underdot{u}$, we find $\dot{T}$ is composed of fourth-order moments of the velocity plus third-order moments involving the velocity (alone and with the force). Then we suppose that $|u^{(2)}| \ll |u^{(1)}|$, and retain in the fourth-order moment only $\underdot{u}^{(1)}$ terms to find [correct to terms $O(u^{(2)^2})$]

$$\dot{T}(k) = 0 \cong \dot{T}^{(1)}(k) + \dot{T}^{(v)}(k) + \dot{T}^{(s)}(k) \qquad (2.12)$$

where $\dot{T}^{(v)}$ stands for the terms proportional to ν; we have

$$\dot{T}^{(1)}(k) = \frac{\pi}{4} k^2 \int \frac{Q}{k^2 k'^2 k''^2} \left\{ 2\left[k^2(k''^2 - 3k'^2) - (k''^2 - k'^2)^2 \right] \dot{\phi}^{(1)}(k,k',k'') \right.$$

$$\left. + 2k''^2(k''^2 - k^2) \dot{\phi}^{(1)}(k',k'',k) \right\} d\underdot{k}'$$

$$(2.13)$$

where

$$\dot{\phi}^{(1)}(k,k',k'') = \frac{E_1(k'')}{16\pi^2 k''^2} \left[\frac{E_1(k')}{k'^2} - \frac{E_1(k)}{k^2} \right] \qquad (2.14)$$

and

$$Q = k^4 + k'^4 + k''^4 - 2k^2 k'^2 - 2k'^2 k''^2 - 2k''^2 k^2 \qquad (2.15)$$

$$\underdot{k} + \underdot{k}' + \underdot{k}'' = 0 \qquad (2.16)$$

The time derivative, $\dot{T}^{(1)} + \dot{T}^{(v)}$, of the transfer is exactly that obtained by Proudman and Reid[15] using the zero-fourth-cumulant hypothesis. (We employ a notation similar to that of Ref. 15 here. The viscous term, $\dot{T}^{(v)}$, will not be used here.)

In (2.13), E_1 is the "Gaussian" part of the energy. It is obtained from (2.7),

$$E_1(k) = \lceil 2\pi^2 V \rceil^{-1} k^2 \langle u_i^{(1)}(\underline{k}) u_i^{(1)}(-\underline{k}) \rangle. \qquad (2.17)$$

From (2.2) and (2.3) using the transform we have at the initial instant $t = 0$,

$$E_1(k,0) = 2(2\pi)^{-2} k^2 \int |K^{(1)}(k; 0,t')|^2 dt' \qquad (2.18)$$

with $K^{(1)}$ defined implicitly by

$$K_{ij}^{(1)}(\underline{k}; 0,t') = P_{ij}(\underline{k}) K^{(1)}(k; 0,t') \qquad (2.19)$$

and with [see (2.3)]

$$K_{ij}^{(1)}(\underline{k}; 0,t') = \int e^{i\underline{k}\cdot\underline{r}} K_{ij}^{(1)}(\underline{r}; 0,t') d\underline{r}. \qquad (2.20)$$

To repeat, it is understood that the representation will change at a later time; we use the representation at $t = 0$.

The source contribution is

$$\dot{T}^{(S)} = \frac{i}{2V} (2\pi)^{-5} k^2 P_{i\alpha\beta}(\underline{k}) \int d\underline{k}' \langle 2u_\alpha(\underline{k} - \underline{k}') f_\beta(k') u_i(-\underline{k})$$

$$+ u_\alpha(\underline{k} - \underline{k}') u_\beta(\underline{k}') f_i(-\underline{k}) \rangle + c.c.$$

$$(2.21)$$

To lowest order each term for $\dot{T}^{(S)}$ initially is of order $u^{(2)}$ just as for the transfer (2.11).

There are important differences between the work in Ref. 15 and the present discussion. The process here is statistically stationary (and of course driven). The equations can be used only at or near a nearly normal statistical equilibrium. In order to follow the approach

to equilibrium, we should have to integrate the equations for the kernels in the nearly normal expansion in the usual way.[5,6] If the initial process is near enough to its equilibrium form, this can be done with a rapidly converging series. However, if the initial process is far from equilibrium, the intermediate stages may be highly non-Gaussian, and the series poorly convergent. Then the Gaussian part of the derivative of the transfer, $\dot{T}^{(1)}$ of (2.13), will not be a good approximation and the expansion fails. This has occurred in some previous computations making use of the zero-fourth-cumulant approximation,[16,18] where (nonphysical) negative energy spectra have been found.

To determine $\dot{T}^{(v)}$, we note that we need the Fourier transforms of the triple velocity correlation.[15] A knowledge of the non-Gaussian term $u^{(2)}$ in the viscous range is needed for their evaluation. We shall examine only the energy and inertial ranges in this paper.

The time derivatives of all moments vanish for stationary processes. One may wonder why we focus attention on $\dot{T}(k)$ [see (2.12)]. In the inertial range the derivatives of <u>even</u> moments are small for nearly normal processes since those derivatives involve $\underline{u}^{(2)}$. The derivative, $\dot{T}$, involving the first important odd moment is chosen because of its central, physical role.

III. INERTIAL RANGE EQUILIBRIUM

At equilibrium we have in the inertial range [see (2.13)], assuming we can separate the viscous, inertial, and energy ranges in the usual way,

$$0 \approx \dot{T}^{(1)}(k) = \frac{\pi}{4} k^2 \int \frac{Q}{k^2 k'^2 k''^2} \left\{ 2 \left[k^2 (k''^2 - 3k'^2) - (k''^2 - k'^2)^2 \right] \dot{\phi}(k,k',k'') \right.$$

$$\left. + 2k''^2 (k''^2 - k^2) \dot{\phi}(k',k'',k) \right\} d\underline{k}' \tag{3.1}$$

with

$$\dot{\phi}(k,k',k'') \approx \frac{E_1(k'')}{16\pi^2 k''^2} \left[\frac{E_1(k')}{k'^2} - \frac{E_1(k)}{k^2} \right] \tag{3.2}$$

for $k_0 < k < k_v$, and k_v is that wavenumber at which viscous effects are no longer negligible.

We continue by solving for the spectrum E_1, which will cause the derivative of the transfer to vanish in the inertial range. Note first the interesting result that the "equipartition" form $E \sim k^2$ is an (exactly Gaussian) solution. Look for an algebraic form, $E_1 = (c/2\pi^2)k^{-n}$. Substitute in (3.1) - (3.2), make changes of variable, and find

$$0 = I_1 - I_2 - I_3 \tag{3.3}$$

with

$$I_j = \int_{a_j}^{b_j} r^{-n} dr \int_{-1}^{c_j} \frac{(1 - u^2) P_j(r,u)}{\left(1 - 2ru + r^2 \right)^{\alpha_j}} du, \qquad j = 1,2,3 \tag{3.4}$$

with quantities given in Table 1.

Table 1

QUANTITIES IN (3.4)

j	a_j	b_j	c_j	$P_j(r,\mu)$	α_j
1	0	∞	1	$1 - 2r\mu + r^3\mu$	1
2	0	$\frac{1}{2}$	1	$1 + r^2 - 3r\mu + 2r^2\mu^2$	$\frac{n}{2} + 2$
3	$\frac{1}{2}$	∞	$\frac{1}{2}r$	$1 + r^2 - 3r\mu + 2r^2\mu^2$	$\frac{n}{2} + 2$

We look for the root of (3.3); the right side diverges (with opposite sign) for $n \to 1^+$, 3^-. A quantity proportion to $\dot{T}$ -- the right side of (3.3) -- is plotted in Fig. 1. The calculation is complicated by cancelling singularities in (3.4). The root is clearly very near to $n = 2$, if not exactly that value. We have not calculated the dashed portion of the curve in Fig. 1; we merely sketch those divergent portions.

It is known (see Ref. 3) that, for a one-dimensional process, k^{-2} is the spectrum which is characteristic of near-discontinuities in the flow. It is easy to show that this is also the case for some three-dimensional flows (see Appendix). Real turbulent flows certainly exhibit near-discontinuities and might be supposed to have an energy spectrum range with a k^{-2} behavior. Indeed early measurements showed approximately this behavior, as can be seen in Fig. 2.

From (3.1) we see we cannot determine the coefficient of k^{-2}. Guided by earlier experimental works (Fig. 2 and Ref. 14, Fig. 7.8) for the energy range we write

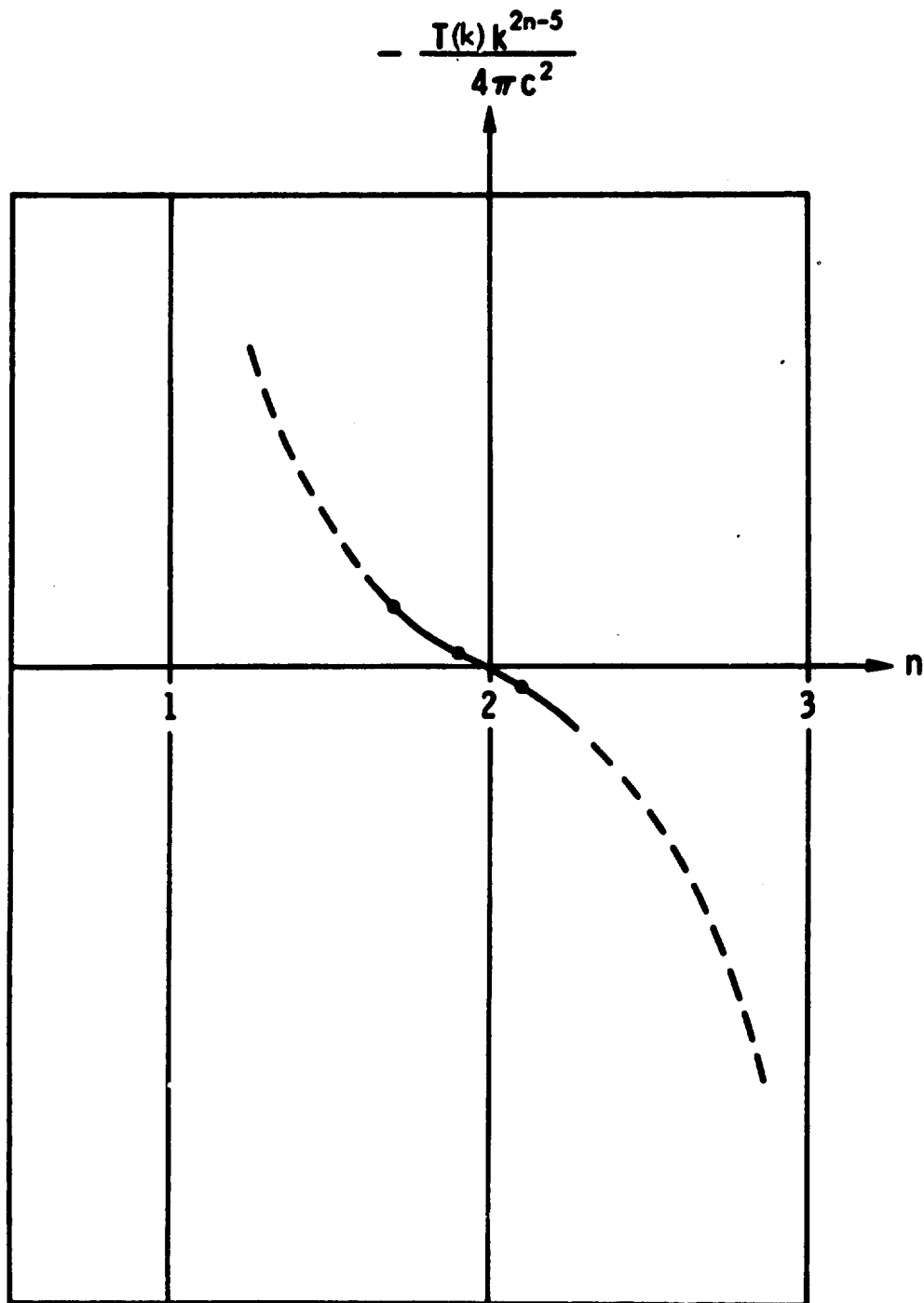$$E(k) = C \frac{u'^2}{\ell} k^{-2} \tag{3.5}$$

Fig. 1 -- Plot of calculation for the equilibrium
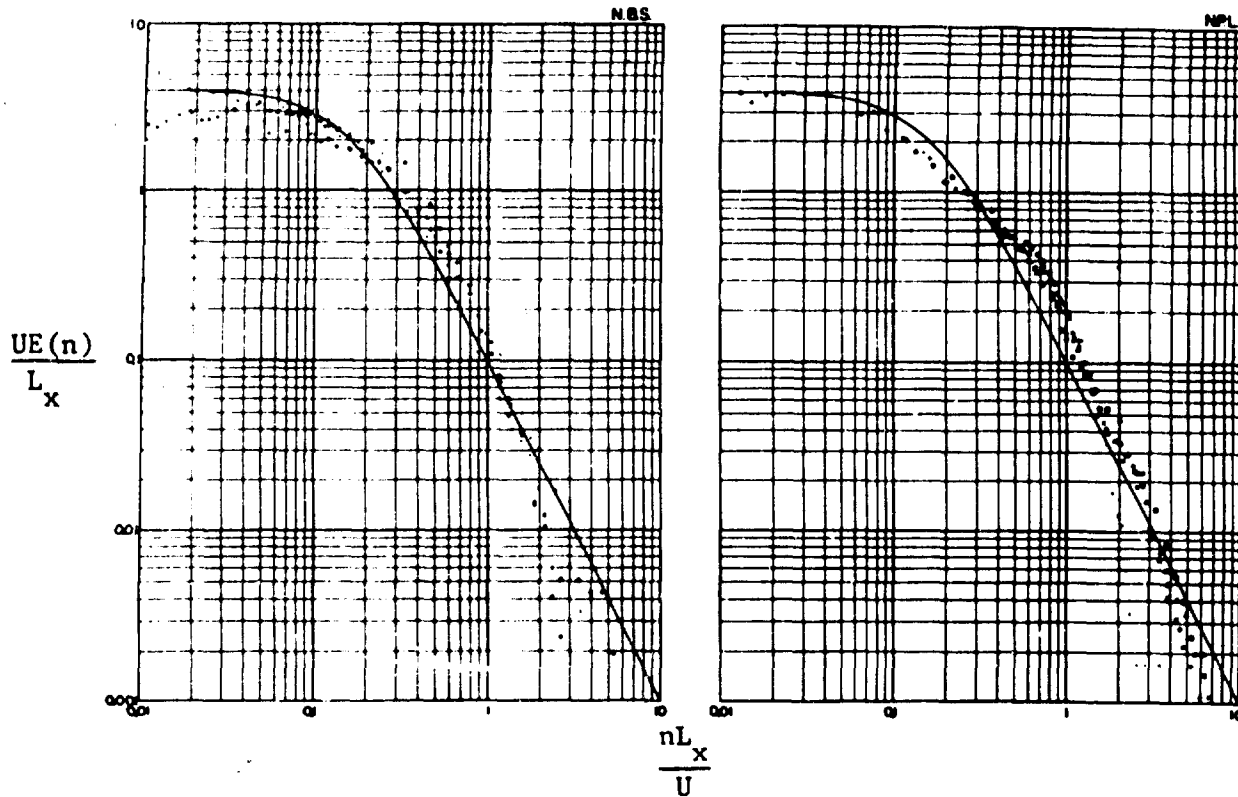spectrum -- the root of Eq. (3.3).

Fig. 2 -- Energy spectra from Dryden, Ref. 13.

Comparison of National Bureau of Standards and National Physical Laboratory measurements of the spectrum of turbulence, plotted nondimensionally.

At left, NBS values 40 (·) and 160 (+) inches behind 1-inch mesh screen at U = 40 ft/sec.

At right, NPL values 82 inches behind 3-inch mesh screen at 15 (·), 20 (x), 25 (+), 30 (△), and 35 ft/sec.

The reference curve in each case is the curve

$$\frac{UF(n)}{L_x} = \frac{4}{1 + \dfrac{4\pi^2 n^2 L_x^2}{U^2}} \ ;$$

U is the mean speed, L_x is the integral $\displaystyle\int_o^\infty R_x \, dx$, R_x is the correlation between the fluctuations at two points separated by the distance x in the direction of flow, n is the frequency and E(n) is the fraction of the total energy of the turbulence arising from frequencies between n and n + dn.

$$\dot{\epsilon} = \int_{\substack{\text{Energy} \\ \text{Range}}} \dot{T}(k)dk \qquad (3.6)$$

with

$$\epsilon = \int \dot{E}(k)dk \qquad (3.7)$$

The Kolmogoroff spectrum $\left(k^{-5/3}\right)$ may depend upon higher-order terms in the C-M-W series.

IV. MIXING-LENGTH CORRECTION

To find $T(k)$ for the statistically stationary process given in (2.11) we see from (2.3) that we shall need a knowledge of $K^{(2)}$ given in terms of the representation at $t = 0$, i.e., in terms of $_{t=0}H_\alpha^{(1)}$. In calculating for turbulence experiments,[5] good results have been obtained using a hypothesis for this kernel (stated in terms of the Fourier transform)

$$K_{ijk}^{(2)}(\underline{k}_1,\underline{k}_2) = \frac{1}{2} t_0 P_{i\gamma\delta}(\underline{k}_1 + \underline{k}_2)K_{\gamma j}^{(1)}(\underline{k}_1)K_{\delta k}^{(1)}(\underline{k}_2). \qquad (4.1)$$

Incidentally this is the form of $K^{(2)}$ for short times if the process is initially exactly Gaussian [see Ref. 5, Eq. (3.14)]. Here t_0 is a characteristic time of order the correlation time; in Ref. 5 t_0 (there called a_0) was found to be 0.4 for some turbulence experiments. We substitute (4.1) in (2.11) and use averages of products of processes (see Ref. 5) and (2.18) to find

$$T(k) = (4\pi)^{-1} t_0 k^2 \left\{ 2 \int \frac{Q(k,k',k'')}{k^2 k'^2 k''^2} \left[k^4 - k'^2(\underline{k}\cdot\underline{k}') + 2k^2(\underline{k}\cdot\underline{k}') \right] \right.$$

$$\times \frac{E(k)}{k^2} \frac{E(k')}{k'^2} d\underline{k}'$$

$$- \int \frac{Q(k,k',k'')}{k^2 k'^2 k''^2} \left[k^4 - k^2(\underline{k}'\cdot\underline{k}'') - 2(\underline{k}\cdot\underline{k}')(\underline{k}\cdot\underline{k}'') \right]$$

$$\left. \times \frac{E(k')}{k'^2} \frac{E(k'')}{k''^2} d\underline{k}' \right\}. \qquad (4.2)$$

We use the definitions of Section II and assume that the energy spectrum is primarily given by its Gaussian part [here just $E(k)$] at least for moderate wavenumbers.

It is interesting that this transfer is, except for a coefficient, $\underline{\text{equal}}$ to $\dot{T}^{(1)}$, the part of $\dot{T}$ using lowest order (Gaussian)

contributions [see (2.12) through (2.15)]. In the inertial range,
$E \sim k^{-2}$ gives zero transfer using (4.1) -- as, in the previous section
it gave zero $\dot{T}$. Thus (4.1) is one solution for $K^{(2)}$ giving $T = 0$ in
the inertial range. The same correspondence held for Burger's model
(see Refs. 3 and 6). We are interested in this section in E and T
for small k in the energy range.

Consider a version of the mixing-length theory. One can define
an eddy viscosity to account for the nonlinear transfer [refer to (2.8)],
treating the effect of that transfer on the large eddies,

$$2\nu_D(k)k^2E(k) \equiv -T(k) \tag{4.3}$$

or

$$\nu_D(k) = -T(k)/[2k^2E(k)] . \tag{4.4}$$

It is sometimes convenient to use (4.3) with ν_D approximated, in
place of the full, nonlinear problem. We expand the expression for
ν_D about k = 0 using the functions above.

To lowest order, the angular integrations in (4.2) for T(k) can
be performed. The result of these manipulations is

$$\nu_D(0) = \frac{2}{15} t_0 \left\{ 2E_T - \frac{7}{E''(0)} \int_0^\infty \frac{E^2(k')}{k'^2} dk' \right\} . \tag{4.5}$$

We suppose that the curvature of E(k) doesn't vanish at the origin for
the nonhomogeneous turbulence being approximated here. Also we define
the turbulent energy per unit mass,

$$E_T = \int_0^\infty E(k')dk' \tag{4.6}$$

The second term in (4.5) tends to emphasize the larger-scale part of the process, reducing the eddy viscosity. The eddy viscosity given by (4.5) is clearly of the order of the usually assumed value, for we have

$$E_T \sim u'^2$$

with u' the fluctuation velocity, so [from (4.5)]

$$\nu_D \sim t_0 E_T$$

$$\sim u'\ell$$

(4.7)

with ℓ the turbulence scale. In the theory under discussion, it is not evident from the start that a form like (4.5) would be obtained at $k = 0$. The result is felt to be encouraging for the use of the nearly normal expansion.

We now continue to the next order term in k. To simplify the discussion we use an estimate for $E(k)$. This restriction isn't necessary but is convenient. Set

$$E(k) = \frac{1}{2} E''(0)k^2 \qquad k < k_0$$

$$= \frac{6}{11} E_T E^{2/3} k^{-5/3} \qquad k > k_0$$

(4.8)

where k_0 is the position of the maximum of E, and E is assumed to be continuous there. The larger wavenumber behavior is of course the Kolmogoroff spectrum. The result, to lower order, for (4.4) is

$$\nu_D(k) \approx t_0 E_T \left[0.123 + 0.139 \left(\frac{k}{k_0} \right)^2 \right]$$

(4.9)

$$\equiv \nu_D(0) + \frac{1}{2} \nu_D''(0)k^2$$

Use of the spectrum (4.8) permits us to calculate all integrals involved. We can plausibly replace the nonlinear terms in the Navier-Stokes equation by $\nu_D(k)k^2\underline{u}$. In physical space, the modified Navier-Stokes equation is

$$\rho\,\frac{\partial \underline{u}}{\partial t} = \left\{\nu_D(0) - \frac{1}{2}\,\nu_D''(0)\nabla^2 + \nu\right\}\,\nabla^2\underline{u}\,,\qquad (4.10)$$

where in turbulent flows ν is negligible. The coefficients are found from (4.9). (It is interesting that the two coefficients in (4.9) are of the same order.) The relation provides a one-parameter form for the two coefficients (remembering that k_0 is the position of the spectral maximum). An experimental check would be helpful.

V. CONCLUSIONS

We have considered the use of the Cameron-Martin-Wiener representation for statistically stationary (and inhomogeneous) turbulence. Difficulties with the time transformation of the white-noise process are avoided by using the representation only at one instant. By dealing with the time derivative of the energy-transfer term, we obtain a formulation similar to the zero-fourth-cumulant approximation -- though differing from it in important ways. From this formulation, a k^{-2} energy spectrum is obtained and discussed. Using a hypothesis, supported by previous computations, a useful correction to mixing-length theory is found.

Appendix

FLOW REALIZATIONS WITH $E \sim k^{-2}$

Some velocity flows with near-discontinuities have spectra with large wavenumber behavior like k^{-2}. Townsend[17] has shown this spectrum results for flows with random shear discontinuities, i.e., vortex sheets. Consider a flow made up of spheres rotating with angular velocity ω. We examine first the Fourier transform of one such sphere:

$$\underline{u}(k) = \iiint e^{i\underline{k}\cdot\underline{r}} \, \underline{u}(\underline{r})d\underline{r} \qquad (A.1)$$

with (in spherical coordinates)

$$\underline{u}(\underline{r}) = \hat{\phi}\omega r \sin\theta, \quad r < a$$
$$= 0, \qquad r > a \qquad (A.2)$$

where $\hat{\phi}$ is the unit azimuthal vector. Let us consider a typical value, e.g., $u_y(k\hat{i})$. Using the method of stationary phase it is easy to show that for $ka \gg 1$,

$$u_y(k\hat{i}) \sim \frac{\omega a^2}{k^2} . \qquad (A.3)$$

Suppose we have a random set of N such spheres in a finite volume V. The energy spectrum[14] is found (using volume averaging)

$$E(k) \sim \frac{k^2 N \underline{u}^2(k)}{V} \qquad (A.4)$$

From (A.3) we see a typical value

$$E(k) \sim \frac{N\omega^2 a^4}{V} k^{-2} \qquad (A.5)$$

It is easy to show many other discontinuous flows with this spectral behavior. It is known that the corresponding behavior holds for nearly discontinuous, one-dimensional time series.[3]

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